

A Generalization of Injective and Projective Complexes

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Abstract

In this paper it is given a generalization of projective and injective complexes.

1 Introduction

Definition 1.1 (\mathcal{X} - complex) Let \mathcal{X} be a class of R -modules. A complex $\mathcal{C} : \dots \rightarrow C^{n-1} \rightarrow C^n \rightarrow C^{n+1} \rightarrow \dots$ is called an \mathcal{X} - (cochain) complex, if $C^i \in \mathcal{X}$ for all $i \in \mathbb{Z}$. A complex $\mathcal{C} : \dots \rightarrow C_{n+1} \rightarrow C_n \rightarrow C_{n-1} \rightarrow \dots$ is called an \mathcal{X} - (chain) complex, if $C_i \in \mathcal{X}$ for all $i \in \mathbb{Z}$. The class of all \mathcal{X} - complexes is denoted by $C(\mathcal{X}^*)$. ($\mathcal{X}^* = \mathcal{X}$ - complex).

Definition 1.2 (\mathcal{X} - injective and \mathcal{X} - projective module) A left R -module E is \mathcal{X} - injective, if $\text{Ext}^1(B/A, E) = 0$ where for every module $B/A \in \mathcal{X}$, whenever i is an injection and f is any map, there

exists a map g making the following diagram commute;

$$\begin{array}{ccccc}
 & & E & & \\
 & & \uparrow & \nearrow & \\
 & f & & g & \\
 0 & \longrightarrow & A & \xrightarrow{i} & B
 \end{array}$$

Let a left R -module P is projective if, $\text{Ext}^1(P, X) = 0$ for every module $X \in \mathcal{X}$ where $X = \text{Ker } p$, whenever p is surjective and h is any map, there exists a map g making the following diagram commute;

$$\begin{array}{ccccc}
 & & P & & \\
 & & \downarrow & \nearrow & \\
 & g & & h & \\
 A & \xrightarrow{p} & B & \longrightarrow & 0
 \end{array}$$

Definition 1.3 (\mathcal{X} -precover, \mathcal{X} -preenvelope) Let \mathcal{A} be an abelian category and let \mathcal{X} be a class of objects of \mathcal{A} . Then for an object $M \in \mathcal{A}$, a morphism $\phi : X \longrightarrow M$ where $X \in \mathcal{X}$ is called an \mathcal{X} -precover of M , if $f : X' \longrightarrow M$ where $X' \in \mathcal{X}$ and the following diagram;

$$\begin{array}{ccc}
 X' & \xrightarrow{f} & M \\
 \downarrow g & \nearrow \phi & \\
 X & &
 \end{array}$$

can be completed to a commutative diagram.

Let \mathcal{A} be an abelian category and let \mathcal{X} be a class of objects of \mathcal{A} . Then for an object $M \in \mathcal{A}$ a morphism $\phi : M \longrightarrow X$ where $X \in \mathcal{X}$ is called an \mathcal{X} -preenvelope of M , if $f : M \longrightarrow X'$ where $X' \in \mathcal{X}$ and the following diagram;

$$\begin{array}{ccc} M & \xrightarrow{f} & X' \\ \phi \downarrow & \nearrow g & \\ & X & \end{array}$$

can be completed to a commutative diagram.

Every complex has an injective and projective resolution. In this paper Ext is defined the same as the class of R -modules.(See [3])

2 \mathcal{X} – injective and \mathcal{X} – projective complexes

Definition 2.1 (\mathcal{X} –injective and \mathcal{X} –projective complexes) A complex \mathcal{C} is called an \mathcal{X} – injective complex, if $\text{Ext}^1(Y/X, C) = 0$ where for every complex $Y/X \in C(\mathcal{X})$. Thus the following diagram commutes as follows;

$$\begin{array}{ccccc}
& & C & & \\
& & \uparrow f & \nearrow \tilde{f} & \\
0 & \longrightarrow & X & \xrightarrow{\phi} & Y
\end{array}$$

such that $\tilde{f}\phi = f$ where ϕ is a monomorphism.

Dually we can define an \mathcal{X} – projective complex. A complex \mathcal{C} is called an \mathcal{X} – projective complex, if $\text{Ext}^1(\mathcal{C}, X) = 0$ for every complex $X \in \mathcal{C}(\mathcal{X})$ where $\text{Ker}\phi = X \in \mathcal{C}(\mathcal{X})$. By diagram we can show as follows;

$$\begin{array}{ccccc}
& & C & & \\
& \nwarrow \tilde{f} & \downarrow f & & \\
A & \xrightarrow{\phi} & B & \longrightarrow & 0
\end{array}$$

such that $\phi\tilde{f} = f$ where ϕ is onto.

Example 2.2 If P is an \mathcal{X} – projective(\mathcal{X} – injective) module, then $\overline{P} : \dots \longrightarrow 0 \longrightarrow P \longrightarrow P \longrightarrow 0 \longrightarrow 0 \longrightarrow \dots$ is an \mathcal{X} – projective(\mathcal{X} – injective) complex. Also the finite direct sum of \mathcal{X} – projective(\mathcal{X} – injective) complexes is again \mathcal{X} – projective(\mathcal{X} – injective).

Note that if P is an \mathcal{X} – injective(\mathcal{X} – projective) module and P is not in the class \mathcal{X} , then \overline{P} is an \mathcal{X} – injective complex, but not an \mathcal{X} – complex. So \mathcal{X} – injective complex may not be an \mathcal{X} – complex.

Proof Consider the following diagram;

$$\begin{array}{ccccc}
& & P & & \\
& \nearrow h_k \cdots & \downarrow f_k & \searrow 1_P & \\
C_k & \xrightarrow{g_k} & C'_k & & P \\
& \searrow d_k & \nearrow h_{k-1} \cdots & \downarrow f_{k-1} & \\
& & C_{k-1} & \xrightarrow{g_{k-1}} & C'_{k-1}
\end{array}$$

where $g : C \rightarrow C' \rightarrow 0$ is an epic morphism such that $\text{Kerg} \in \mathcal{X}$ and $f : \overline{P} \rightarrow C$ is a morphism and d, d' are differentials of C, C' , respectively. Since P is an \mathcal{X} -projective module, there exists a morphism $h_k : P \rightarrow C_k$ with $g_k h_k = f_k$.

Define $h_{k-1} : P \rightarrow C_{k-1}$ with $h_{k-1} = d_k h_k$. Then $g_{k-1} h_{k-1} = f_{k-1}$. Thus we have a chain map $h : \overline{P} \rightarrow C$, as required. Dually, we can prove that if P is an \mathcal{X} -injective module, then \overline{P} is an \mathcal{X} -injective complex. \square

Definition 2.3 (*DG(\mathcal{X} -injective) and DG(\mathcal{X} -projective) complexes*)

Let ε be the class of exact complexes. A complex I is called DG(\mathcal{X} -injective), if each I^n is \mathcal{X} -injective and $\mathcal{H}om(E, I)$ is exact for all $E \in \varepsilon$ where $d^n : E^n \rightarrow E^{n+1}$ with $\text{Ker} d^n \in \mathcal{X}$.

A complex I is called DG(\mathcal{X} -projective), if each I^n is \mathcal{X} -projective and $\mathcal{H}om(I, E)$ is exact for all $E \in \varepsilon$ where $d^n : E^n \rightarrow E^{n+1}$ with $\text{Ker} d^n \in \mathcal{X}$.

Lemma 2.4 Let $A \xrightarrow{\beta} B \xrightarrow{\theta} C$ be an exact sequence where $\text{Ker}\theta \in \mathcal{X}$. Then for all \mathcal{X} -projective complex I ,

$$\text{Hom}(I, A) \longrightarrow \text{Hom}(I, B) \longrightarrow \text{Hom}(I, C)$$

is exact.

Proof $0 \longrightarrow \text{Ker}\theta \xrightarrow{\beta} B \xrightarrow{\theta} C$ is exact, so

$$0 \longrightarrow \text{Hom}(I, \text{Ker}\theta) \longrightarrow \text{Hom}(I, B) \longrightarrow \text{Hom}(I, C)$$

is exact.

We have the following commutative diagram;

$$\begin{array}{ccccc} A & \xrightarrow{\beta} & \text{Im}\beta & \longrightarrow & 0 \\ & \searrow f & \uparrow g & & \\ & & I & & \end{array}$$

such that $\beta f = g$.

Since I is \mathcal{X} -projective and $\text{Ker}\theta \in \mathcal{X}$.

Therefore, $\text{Hom}(I, A) \longrightarrow \text{Hom}(I, B) \longrightarrow \text{Hom}(I, C)$ is exact. \square

Dually we can give the following lemma;

Lemma 2.5 Let $A \xrightarrow{\beta} B \xrightarrow{\theta} C$ be an exact sequence where $\frac{C}{\text{Im}\theta} \in \mathcal{X}$. Then for all \mathcal{X} -injective complex I ,

$$\text{Hom}(C, I) \longrightarrow \text{Hom}(B, I) \longrightarrow \text{Hom}(A, I)$$

is exact.

Proof It is clear from Lemma 2.4. \square

Example 2.6 $(DG(\mathcal{X} - \text{injective})(DG(\mathcal{X} - \text{projective})))$ Let $I = \dots \longrightarrow 0 \longrightarrow I^0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \dots$ where I^0 is an \mathcal{X} -injective(\mathcal{X} - projective) module. Then I is $DG(\mathcal{X} - \text{injective})(DG(\mathcal{X} - \text{projective}))$ complex.

Proof Let

$$E : \dots \longrightarrow E^{-1} \xrightarrow{d^{-1}} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \xrightarrow{d^2} E^3 \longrightarrow \dots$$

is exact and $Ker d^n \in \mathcal{X}$, then

$$\mathcal{H}om(E, I) \cong \dots \longrightarrow \mathcal{H}om(E^2, I^0) \longrightarrow \mathcal{H}om(E^1, I^0) \longrightarrow \mathcal{H}om(E^0, I^0) \longrightarrow \dots$$

.

By Lemma 2.4 $\mathcal{H}om(E, I)$ is exact. \square

Lemma 2.7 If a complex $X : \dots \longrightarrow X_{n+1} \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \dots$ is an \mathcal{X} - injective(\mathcal{X} - projective) complex, then for all $n \in \mathbb{Z}$ X_n is an \mathcal{X} - injective(\mathcal{X} - projective) module.

Proof Let $0 \longrightarrow N \xrightarrow{i} M$ and $\frac{M}{N} \in \mathcal{X}$ and $\alpha : N \rightarrow X_n$ be linear form the pushout;

$$\begin{array}{ccc}
N & \xrightarrow{i} & M \\
\downarrow \alpha & & \downarrow \gamma_n \\
X_n & \xrightarrow{\theta_n} & \frac{X_n \oplus M}{A}
\end{array}$$

where $A = \{(\alpha(n), -i(n)) : n \in N\}$.

By the following diagram;

$$\begin{array}{ccccccc}
0 & \longrightarrow & X_{n+1} & \longrightarrow & X_{n+1} & \longrightarrow & 0 \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & X_n & \longrightarrow & \frac{M \oplus X_n}{A} & \longrightarrow & \frac{M}{N} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & X_{n-1} & \longrightarrow & X_{n-1} & \longrightarrow & 0 \longrightarrow 0
\end{array}$$

We have the exact sequence $0 \longrightarrow X \longrightarrow T \longrightarrow S \longrightarrow 0$ where

$$T : \dots \longrightarrow X_{n+2} \longrightarrow X_{n+1} \longrightarrow \frac{M \oplus X_n}{A} \longrightarrow X_{n-1} \dots \text{ and}$$

$$S : \dots \longrightarrow 0 \longrightarrow 0 \longrightarrow \frac{M}{N} \longrightarrow 0 \dots$$

Since X is a \mathcal{X} -injective complex, $Ext^1(S, X) = 0$, and so

$$0 \rightarrow Hom(S, X) \rightarrow Hom(T, X) \rightarrow Hom(X, X) \rightarrow Ext^1(S, X) =$$

0. Therefore there exists $\beta_n : T_n = \frac{M \oplus X_n}{A} \longrightarrow X_n$ such that $\beta_n \theta_n = 1$.

So

$$\beta^n \theta^n(\alpha(n)) = \alpha(n)$$

$$\beta^n((\alpha(n), 0) + A) = \alpha(n)$$

$$\beta^n((0, i) + A) = \alpha(n)$$

$$\beta^n \gamma_n i(n) = \alpha(n)$$

And hence $\beta^n \gamma_n i = \alpha$. So X_n is a \mathcal{X} -injective module. \square

Remark 2.8 Let $X : \dots \rightarrow X_{n+1} \rightarrow X_n \rightarrow X_{n-1} \rightarrow \dots$ be a complex such that X_n are \mathcal{X} -injective (\mathcal{X} -projective) modules for all $n \in \mathbb{Z}$. It need not be that X is an \mathcal{X} -injective (\mathcal{X} -projective) complex.

Let R be an injective module and $f : R \rightarrow R$ be a $1 - 1$ morphism. Then we can find $g \neq 0$ and $g : R \rightarrow R$ such that $gf = 0$. We have the following diagram;

$$\begin{array}{ccccccccc}
 \dots & \longrightarrow & 0 & \longrightarrow & R & \xrightarrow{f} & R & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow (f,0) & & \downarrow 1 & & \downarrow & & \\
 \dots & \longrightarrow & 0 & \longrightarrow & R \oplus S & \xrightarrow{(i,0)} & R & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow g & & \downarrow & & \\
 \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & R & \longrightarrow & 0 & \longrightarrow & \dots
 \end{array}$$

Then $g(i, 0) \neq 0$. So \overline{R} cannot be an \mathcal{X} -injective complex.

Dually, we can give an example for \mathcal{X} -projective.

Definition 2.9 Let ε be the class of exact complexes. Then we can define ε_1 such that ε_1 is the class of exact complexes whose kernels are in \mathcal{X} .

Lemma 2.10 Let I be an \mathcal{X} -projective(injective) module. Then \underline{I} is in $\varepsilon_1^\perp({}^\perp\varepsilon_1)$.

Proof Let

$$0 \longrightarrow I \longrightarrow I^1 \xrightarrow{\lambda_1} I^2 \xrightarrow{\lambda_2} I^3 \longrightarrow \dots$$

be a projective resolution of I .

Then $Hom(E, I^1) \xrightarrow{\alpha} Hom(E, I^2) \xrightarrow{\beta} Hom(E, I^3)$ where $E : \dots \rightarrow E^{-2} \rightarrow E^{-1} \rightarrow E^0 \rightarrow E^1 \rightarrow \dots$

$$Ext^1(E, I) = ?$$

$$\begin{array}{ccccccc}
E : \dots & \longrightarrow & E^{-3} & \xrightarrow{d^{-3}} & E^{-2} & \xrightarrow{d^{-2}} & E^{-1} & \xrightarrow{d^{-1}} & E^0 & \longrightarrow & \dots \\
& & \downarrow & & \downarrow u^{-2} & & \downarrow u^{-1} & & \downarrow & & \\
I^2 : \dots & \longrightarrow & 0 & \longrightarrow & I^0 & \xrightarrow{1} & I^0 & \longrightarrow & 0 & \longrightarrow & \dots \\
& & \downarrow & & \downarrow 1 & & \downarrow & & \downarrow & & \\
I^3 : \dots & \longrightarrow & I^0 & \longrightarrow & I^0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots
\end{array}$$

where $\lambda_2^{-2}u^{-2} = 0$, since $\lambda_2^{-2} = 1$, then $u^{-2} = 0$ and also since $u^{-2} = u^{-1}d^{-2}$, then $0 = u^{-1}d^{-2}$.

$$\begin{array}{ccccccccccc}
E : \dots & \longrightarrow & E^{-3} & \xrightarrow{d^{-3}} & E^{-2} & \xrightarrow{d^{-2}} & E^{-1} & \xrightarrow{d^{-1}} & E^0 & \longrightarrow & E^1 & \longrightarrow & \dots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
I^1 : \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & I^0 & \xrightarrow{1} & I^0 & \longrightarrow & 0 & \longrightarrow & \dots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
I^2 : \dots & \longrightarrow & 0 & \longrightarrow & I^0 & \longrightarrow & I^0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots
\end{array}$$

where $f^0d^{-1} = f^{-1}$ and $f^{-1}d^{-2} = 0$.

Since $\text{Ker}d^{-1} \xrightarrow{i} E^{-1} \xrightarrow{d^{-1}} E^0$ is exact and I^0 is \mathcal{X} -injective, then $\text{Hom}(E^0, I^0) \xrightarrow{(d^{-1})^*} \text{Hom}(E^{-1}, I^0) \xrightarrow{i^*} \text{Hom}(\text{Ker}d^{-1}, I^0)$ is exact.

□

Corollary 2.11 *Let $I = \dots \rightarrow 0 \rightarrow I_n \rightarrow I_{n-1} \rightarrow \dots \rightarrow I_0 \rightarrow 0 \rightarrow \dots$ where I_i is an \mathcal{X} -projective(injective) module, then I is in $\varepsilon_1^\perp({}^\perp\varepsilon_1)$.*

Proof Since ε_1^\perp is extension closed, by Lemma 2.10 we can understand that I is in ε_1^\perp . □

Corollary 2.12 *Every left(right) bounded complex I where I_i is an \mathcal{X} -projective(injective) module is in $\varepsilon_1^\perp({}^\perp\varepsilon_1)$.*

Lemma 2.13 *If $I \in \varepsilon_1^\perp$, then each I^n is \mathcal{X} -injective for each $n \in \mathbb{Z}$.*

Proof Let $S \subseteq M$ be a submodule of a module M with $\frac{M}{S} \in \mathcal{X}$ and $\alpha : S \longrightarrow I_n$ be linear form the pushout;

$$\begin{array}{ccc}
 S & \xrightarrow{i} & M \\
 \alpha \downarrow & & \downarrow i_1 \\
 I^n & \xrightarrow{i_2} & \frac{I^n \oplus M}{A} = I^n \oplus_S M
 \end{array}$$

where $A = \{(\alpha(s), -s) : s \in S\}$. Thus i_2 is one-to-one the same as
i. Then $\bar{I} : \dots \longrightarrow I^{n-1} \longrightarrow I^n \oplus_S M \longrightarrow I^{n+1} \longrightarrow I^{n+2} \longrightarrow \dots$ is a complex.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & I^{n-1} & \longrightarrow & I^{n-1} & \longrightarrow & 0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & I^n & \longrightarrow & I^n \oplus_S M & \longrightarrow & \frac{M}{S} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & I^{n+1} & \longrightarrow & I^{n+1} & \longrightarrow & \frac{M}{S} \longrightarrow 0
 \end{array}$$

Therefore, we have an exact sequence $0 \longrightarrow I \longrightarrow \bar{I} \longrightarrow E \longrightarrow 0$ where $E : \dots \longrightarrow \frac{M}{S} \longrightarrow \frac{M}{S} \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \dots$ and so we have an exact sequence $0 \longrightarrow \text{Hom}(E, I) \longrightarrow \text{Hom}(\bar{I}, I) \longrightarrow \text{Hom}(I, I) \longrightarrow \text{Ext}'(E, I) = 0$ since $I \in \varepsilon_1^\perp$.

This implies that we can find $\bar{f} : \bar{I} \longrightarrow I$ with $\bar{f}f = 1$. Therefore, there exists a function $\bar{f}^n : I^n \oplus_S M \longrightarrow I^n$ with $\bar{f}^n f^n = 1$. So,

$$\bar{f}^n f^n(\alpha(s)) = \alpha(s)$$

$$\bar{f}^n((\alpha(s), 0) + A) = \alpha(s)$$

$$\bar{f}^n((0, s) + A) = \alpha(s)$$

$$\bar{f}^n i_1 i(s) = \alpha(s)$$

and hence $\bar{f}_n i_1 i = \alpha$ and thus each $I^n \in \mathcal{X}$ – *injective*.

□

Lemma 2.14 *Let $f : X \longrightarrow Y$ be a morphism of complexes. Then the exact sequence $0 \longrightarrow Y \longrightarrow M(f) \longrightarrow X[1] \longrightarrow 0$ associated with the mapping cone $M(f)$ splits in $\mathcal{C}(\mathcal{X})$ if and only if f is homotopic to 0.*

Proof It follows from the proof by [3].

□

Lemma 2.15 *Let X and I are complexes. If $\text{Ext}^1(X, I[n]) = 0$ for all $n \in \mathbb{Z}$, then $\mathcal{H}\text{om}(X, I)$ is exact.*

Proof Since $Ext^1(X, I(n)) = 0$, if $f : X[-1] \rightarrow I[n]$ is a morphism, then $0 \rightarrow I[n] \rightarrow M(f) \rightarrow X \rightarrow 0$ splits.

By 2.14, $f : X[-1] \rightarrow I[n]$ is homotopic to zero for all n .

So $f^1 : X \rightarrow I[n+1]$ is homotopic to zero for all $n \in \mathbb{Z}$. Thus $\mathcal{H}om(X, I)$ is exact. \square

Corollary 2.16 $\varepsilon_1^\perp({}^\perp\varepsilon_1) \subseteq DG(\mathcal{X} - \text{projective}(\text{injective}))$.

Proof It follows from Lemma 2.13 and 2.15. \square

Corollary 2.17 Every left(right) bounded complex I where I_i is an \mathcal{X} -projective(injective) module is in $DG(\mathcal{X}$ -projective(injective)).

Proof We can understand by Corollary 2.12 and Corollary 1.16. \square

Lemma 2.18 Let X be extension closed and let I be a $DG(\mathcal{X}$ -injective)($DG(\mathcal{X})$ -projective) complex($\text{proj} \in \mathcal{X}^?$). Then $Ext^1(E, I[n]) = 0$ for all $n \in \mathbb{Z}$ where E is exact and $\text{Ker}\lambda_1 \in \mathcal{X}$ with $\lambda_n : E_n \rightarrow E_{n-1}$.

Proof Let $\dots \rightarrow P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} E \rightarrow 0$ be a projective resolution of E . Then we have

$$\mathcal{H}om(P_0, I) \xrightarrow{f_1^*} \mathcal{H}om(P_1, I) \xrightarrow{f_2^*} \mathcal{H}om(P_2, I)$$

.

Let $\beta \in \text{Ker}f_2^*$. Is $\beta \in \text{Im}f_1^*$?

Since $P_2^n \rightarrow P_1^n \rightarrow P_0^n$ is exact and $E^n \in \mathcal{X}$ and I^n is \mathcal{X} -injective by Lemma 2.6

$$Hom(P_0^n, I^n) \xrightarrow{f_1^{n*}} Hom(P_1^n, I^n) \xrightarrow{f_2^*} Hom(P_2^n, I^n)$$

is exact.

$$f_2^* \beta = 0 \text{ implies that } \beta f_2 = 0$$

$$\Rightarrow \beta^n f_2^n = 0$$

, for all $n \in \mathbb{Z}$

$$\Rightarrow f_2^* \beta^n = 0$$

$$\Rightarrow \beta^n \in Ker f_2^* = Im f_1^*$$

$$\Rightarrow \exists \theta^n \in Hom(P_0^n, I^n)$$

such that

$$\theta^n f_1^n = \beta^n$$

Is θ a chain map?

We know that $Hom(E, I)$ is exact,

(i) Is $Hom(P_0, I)$ exact? (where I is \mathcal{X} -injective and $P_0 \longrightarrow E \longrightarrow 0$ and $E^n \in \mathcal{X}$)

(ii) Let $Hom(P_0, I)$ be exact. We have $\theta^n : P_0^n \longrightarrow I^n$. Is it necessary θ is a chain map?

$$\begin{array}{ccccc}
P_0^{n-1} & \xrightarrow{\lambda^{n-1}} & P_0^n & \xrightarrow{\lambda^n} & P_0^{n+1} \\
\downarrow \theta^{n-1} & \nearrow \cdots & \downarrow \theta^n & \nearrow \cdots & \downarrow \theta^{n+1} \\
I^{n-1} & \xrightarrow{\gamma^{n-1}} & I^n & \xrightarrow{\gamma^n} & I^{n+1}
\end{array}$$

$$\begin{array}{ccccc}
P_0^{n-1} & \xrightarrow{\lambda^{n-1}} & P_0^n & \xrightarrow{\lambda^n} & P_0^{n+1} \\
\downarrow t^{n-1} & \nearrow \cdots s^n & \downarrow t^n & \nearrow \cdots s^{n+1} & \downarrow t^{n+1} \\
I^n & \xrightarrow{\gamma^n} & I^{n+1} & \xrightarrow{\gamma^{n+1}} & I^{n+2}
\end{array}$$

where $t^{n-1} = \theta^n \lambda^{n-1} - \gamma^{n-1} \theta^{n-1}$ and $t^n = \theta^{n+1} \lambda^n - \gamma^n \theta^n$

Since $\mathcal{H}om(P_0, I[n])$ is exact, we have a homotopy such that

$$s^{n+1} \lambda^n + \gamma^n s^n = \theta^{n+1} \lambda^n - \gamma^n \theta^n$$

$$\gamma^n (s^n + \theta^n) = (\theta^{n+1} + s^{n+1}) \lambda^n$$

So we have a chain map. But we investigate a chain map such that

$$\theta^n f_1^n = \beta^n$$

. How can we do?

□

Lemma 2.19 *Let $f : X \longrightarrow Y$ a chain morphism, Y is an \mathcal{X} complex and X is an \mathcal{X} - projective complex. Then f is a homotopic to zero.*

Proof Let $id : Y \longrightarrow Y$ and the exact sequence $0 \longrightarrow Y[-1] \longrightarrow M(id)[-1] \longrightarrow Y \longrightarrow 0$.

Since X is an \mathcal{X} - projective complex, we have the following commutative diagram;

$$\begin{array}{ccccc}
 M(id)[-1] & \xrightarrow{\pi} & Y & \longrightarrow & 0 \\
 \vdots \uparrow g & & \nearrow f & & \\
 X & & & &
 \end{array}$$

where $\pi g = f$. Let $\pi' : M(id)[-1] \longrightarrow Y[-1]$ be a projection. Then if we take an $s = \pi' g$, then for all $n \in \mathbb{Z}$, $s^{n+1}\lambda^n + \gamma^{n-1}s^n = f^n$ where λ and γ are boundary maps of the complexes of X and Y , respectively. So f is homotopic to zero. \square

Lemma 2.20 *Let $f : X \longrightarrow Y$ be a chain morphism, X is an \mathcal{X} - complex and Y is an \mathcal{X} - injective complex. Then f is homotopic to zero.*

Proof Let $id : X \longrightarrow X$, then we have the following exact sequence;

$$\begin{array}{ccccccc}
 0 & \longrightarrow & X & \xrightarrow{i} & M(id) & \longrightarrow & X[1] \longrightarrow 0 \\
 & & \downarrow f & & \nearrow g & & \\
 & & Y & & & &
 \end{array}$$

where $gi = f$.

Let $i' : X[1] \longrightarrow M(id)$ and $s : X[1] \longrightarrow Y$ such that $s = gi'$ with $s_n = g^{n-1}i'$.

$$\begin{array}{ccccc}
X^{n-2} \oplus X^{n-1} & \xrightarrow{u^{n-2}} & X^{n-1} \oplus X^n & \xrightarrow{u^{n-1}} & X^n \oplus X^{n+1} \\
\downarrow g^{n-2} & & \downarrow g^{n-1} & & \downarrow g^n \\
Y^{n-2} & \xrightarrow{\gamma^{n-2}} & Y^{n-1} & \xrightarrow{\gamma^{n-1}} & Y^n
\end{array}$$

$$\begin{aligned}
s^{n+1}\lambda^n + \gamma^{n-1}s^n &= g^n i' \lambda^n + \gamma^{n-1} g^{n-1} i' = g^n (i' \lambda^n + g^n u^{n-1} i' = \\
g^n i' \lambda^n + u^{n-1} i') &= g^n i = f^n
\end{aligned}
\quad \square$$

Proposition 2.21 *Let $C \longrightarrow X$ and $C' \longrightarrow X$ be \mathcal{X} -projective covers of $X \in \mathcal{X}$, then C and C' are homotopic.*

Proof It follows from [4]. \square

Lemma 2.22 *Let $X : \dots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow 0$ be an exact and P be a complex with for all $n \geq 0$, P_n \mathcal{X} -projective module, then $f : P \longrightarrow X$ is a homotopy.*

Proof It follows from [4]. \square

Lemma 2.23 *Let every R -module has an onto \mathcal{X} -projective \mathcal{X} precover with kernel in \mathcal{X} . Then every bounded complex has an $C(\mathcal{X}$ -projective) precover.*

Proof Let $Y(n) : \dots \rightarrow 0 \rightarrow Y^0 \rightarrow Y^1 \rightarrow \dots \rightarrow Y^n \rightarrow 0 \rightarrow \dots$

We will use induction on n . Let $n = 0$, then we have the following commutative diagram;

$$\begin{array}{ccccccccc}
 D(0) : \dots & \longrightarrow & 0 & \longrightarrow & P^0 & \xrightarrow{id} & P^0 & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow f^0 & & \downarrow & & \downarrow & & \\
 Y(0) : \dots & \longrightarrow & 0 & \longrightarrow & Y^0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots
 \end{array}$$

where $D(0)$ is exact and $Ker(D(0) \rightarrow Y(0)) \in \mathcal{X}$.

Let $n = 1$, then we have the following commutative diagram;

$$\begin{array}{ccccccccccccc}
 D(1) : \dots & \longrightarrow & 0 & \longrightarrow & P^0 & \xrightarrow{\lambda_1^0} & P^0 \oplus P^1 & \xrightarrow{\lambda^1} & P^1 & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & & & \downarrow f^0 & & \downarrow (0, f^1) & & \downarrow & & & & \\
 Y(1) : \dots & \longrightarrow & 0 & \longrightarrow & Y^0 & \xrightarrow{a^0} & Y^1 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots
 \end{array}$$

where $D(1)$ is exact and $Ker(D(1) \rightarrow Y(1)) \in \mathcal{X}$.

We also have the following commutative diagram;

$$\begin{array}{ccc}
 P^0 & \xrightarrow{s^0} & P^1 \\
 \downarrow f^0 & & \downarrow f^1 \\
 Y^0 & \xrightarrow{a^0} & Y^1
 \end{array}$$

$$\lambda_1^0(x) = (x, s^0(x)) \text{ and } \lambda^1(x, y) = s^0(x) - y$$

We assume that the following diagram which is commutative;

$$\begin{array}{ccccccccccc}
 D(n) : \dots 0 & \longrightarrow & P^0 & \xrightarrow{\lambda^0} & P^0 \oplus P^1 & \xrightarrow{\lambda^1} & \dots & \longrightarrow & P^{n-1} \oplus P^n & \xrightarrow{\lambda^n} & P^n & \longrightarrow & 0 \\
 & & \downarrow f^0 & & \downarrow (0, f^1) & & & & \downarrow (0, f^n) & & \downarrow & & \\
 Y(n) : \dots 0 & \longrightarrow & Y^0 & \xrightarrow{a^0} & Y^1 & \xrightarrow{a^1} & \dots & \longrightarrow & Y^n & \longrightarrow & 0 & \longrightarrow & 0
 \end{array}$$

where $D(n)$ is exact and $Ker(D(n) \rightarrow Y(n)) \in \mathcal{X}$.

And we also have the following commutative diagrams;

$$\begin{array}{ccc}
D(n) & \xrightarrow{s} & \overline{P^{n+1}} \\
\downarrow & & \downarrow \\
Y(n) & \longrightarrow & \underline{Y^{n+1}}
\end{array}$$

So,

$$\begin{array}{ccccccccccccccc}
D(n) : \dots 0 & \longrightarrow & P^0 & \xrightarrow{\lambda^0} & P^0 \oplus P^1 & \xrightarrow{\lambda^1} & \dots & \xrightarrow{\lambda^{n-1}} & P^{n-1} \oplus P^n & \xrightarrow{\lambda^n} & P^n & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & & & \downarrow s^1 & & \downarrow s^2 & & \\
\overline{P^{n+1}} : \dots 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & P^{n+1} & \xrightarrow{1} & P^{n+1} & \longrightarrow & 0
\end{array}$$

where $s^2 \lambda^n = s^1$ and $s^1 \lambda^{n-1} = 0$.

$$\begin{array}{ccc}
P^{n-1} \oplus P^n & \xrightarrow{s^1} & P^{n+1} \\
\downarrow (0, f^n) & & \downarrow f^{n+1} \\
Y^n & \xrightarrow{a^n} & Y^{n+1}
\end{array}$$

where $f^{n+1} s^1 = a^n(0, f^n)$.

$$\begin{array}{ccc}
P^n & \xrightarrow{s^2} & P^{n+1} \\
\downarrow & & \downarrow f^{n+1} \\
0 & \longrightarrow & Y^{n+1}
\end{array}$$

where $f^{n+1}s^2 = 0$.

$$\begin{array}{ccccccccccc}
D(n+1) : \dots 0 & \longrightarrow & P^0 & \xrightarrow{\lambda^0} & P^0 \oplus P^1 \dots & \longrightarrow & P^{n-1} \oplus P^n & \xrightarrow{\lambda_1^n} & P^n \oplus P^{n+1} & \xrightarrow{\lambda^{n+1}} & P^{n+1} \dots \\
& & \downarrow f^0 & & \downarrow (0, f^1) & & \downarrow (0, f^n) & & \downarrow (0, f^{n+1}) & & \\
Y(n+1) : \dots 0 & \longrightarrow & Y^0 & \xrightarrow{a^0} & Y^1 \dots & \xrightarrow{a^1} & Y^n & \xrightarrow{a^n} & Y^{n+1} & \longrightarrow & \dots
\end{array}$$

where $D(n+1)$ is exact and $\text{Ker}(D(n+1) \rightarrow Y(n+1)) \in \mathcal{X}$,

$$\lambda_1^n(x, y) = (\lambda^n(x, y), s^1(x, y)), \quad \lambda^{n+1}(x, y) = s^2(x) - y.$$

Therefore, $Y(n)$ has an $C(\mathcal{X})$ -projective precover.

□

Lemma 2.24 *Let every R -module has an epic \mathcal{X} -injective \mathcal{X} preenvelope with cokernel in \mathcal{X} . Then every bounded complex has an $C(\mathcal{X} - \text{injective})$ preenvelope.*

Proof Let $Y(n) : \dots \rightarrow 0 \rightarrow Y^0 \rightarrow Y^1 \rightarrow \dots \rightarrow Y^n \rightarrow 0 \rightarrow \dots$

We will use induction on n . Let $n = 0$, then we have the following commutative diagram;

$$\begin{array}{ccccccccccc}
 Y(0) : \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & Y^0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 E(0) : \dots & \longrightarrow & 0 & \longrightarrow & E_0 & \xrightarrow{id} & E_0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots
 \end{array}$$

where $E(0)$ is exact and cokernel in \mathcal{X} .

Let $n = 1$, then we have the following commutative diagram;

$$\begin{array}{ccccccccccc}
 Y(1) : \dots 0 & \longrightarrow & 0 & \longrightarrow & Y_1 & \xrightarrow{a_1} & Y_0 & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & (f_1, 0) & & f_0 & & & & \\
 E(1) : \dots 0 & \longrightarrow & E_1 & \xrightarrow{\lambda_2} & E_1 \oplus E_0 & \xrightarrow{\lambda_1} & E_0 & \longrightarrow & 0 & \longrightarrow & \dots
 \end{array}$$

where $E(1)$ is exact and cokernel in \mathcal{X} .

We also have the following commutative diagram;

$$\begin{array}{ccc} \underline{Y_1} & \longrightarrow & \underline{Y_0} \\ \downarrow & & \downarrow \\ \underline{E_1} & \longrightarrow & \underline{E_0} \end{array}$$

where $E_1 \xrightarrow{\lambda} E_0$ such that $f_0 a_1 = \lambda f_1$.

$$\lambda_2(x) = (x, -f(x)) \text{ and } \lambda_1(x, y) = f(x) + y.$$

We assume that the following diagram which is commutative;

$$\begin{array}{ccccccccccc} Y(n) : \dots 0 & \longrightarrow & 0 & \longrightarrow & Y_n & \xrightarrow{a_n} & \dots & \longrightarrow & Y_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow (f_n, 0) & & & & \downarrow f_0 & & \\ E(n) : \dots 0 & \longrightarrow & E_n & \xrightarrow{\lambda_n} & E_n \oplus E_{n-1} & \xrightarrow{\lambda_{n-1}} & \dots & \longrightarrow & E_0 & \longrightarrow & 0 \end{array}$$

where $E(n)$ is exact and cokernel in \mathcal{X} .

And we have the following commutative diagram;

$$\begin{array}{ccccccc}
Y(n+1) : \dots 0 & \longrightarrow & 0 & \longrightarrow & Y_{n+1} & \xrightarrow{a_{n+1}} & Y_n \dots & \xrightarrow{a_n} & Y_0 & \longrightarrow & \dots \\
& & \downarrow & & \downarrow (f_{n+1}, 0) & & \downarrow (f_n, 0) & & \downarrow f^0 & & \\
E(n+1) : \dots 0 & \longrightarrow & E_{n+1} & \xrightarrow{\lambda_{n+1}} & E_{n+1} \oplus E_n & \xrightarrow{\lambda_n} & E_n \oplus E_{n-1} \dots & \longrightarrow & E_0 & \longrightarrow & \dots
\end{array}$$

where $E(n+1)$ is exact and cokernel in \mathcal{X} .

We also have the following commutative diagrams;

$$\begin{array}{ccc}
\overline{Y_{n+1}} & \longrightarrow & Y(n) \\
\downarrow & & \downarrow \\
\overline{E_{n+1}} & \longrightarrow & E(n)
\end{array}$$

$$\begin{array}{ccc}
Y_{n+1} & \xrightarrow{a_{n+1}} & Y_n \\
\downarrow f_{n+1} & & \downarrow (f_n, 0) \\
E_{n+1} & \xrightarrow{s_{n+1}} & E_n \oplus E_{n-1}
\end{array}$$

where $(f_n, 0)a_{n+1} = s_{n+1}f_{n+1}$.

$$\begin{array}{ccc}
E_{n+1} & \xrightarrow{t_{n+1}} & E_n \\
\downarrow 1 & & \downarrow \lambda_n^1 \\
E_{n+1} & \xrightarrow{s_{n+1}} & E_n \oplus E_{n-1}
\end{array}$$

where $\lambda_n^1 t_{n+1} = s_{n+1} 1$.

Then, $\lambda_{n+1}(x) = (x, -f_{n+1}(x))$, $\lambda_n(x, y) = s_{n+1}(x) + \lambda_n^1(y)$.

Therefore, $Y(n)$ has an $C(\mathcal{X})$ -injective preenvelope.

□

Lemma 2.25 *Let X be an \mathcal{X} -injective complex and $\frac{E(X)}{X} \in \mathcal{X}$ (or $\frac{Y}{X} \in \mathcal{X}$) where $E(X)$ is an injective envelope of X . Then $X = E(X)$ and so it is an exact complex. (X is a direct summand of Y).*

Proof We know that every complex has an injective envelope, so X has an injective envelope $E(X)$. Then $E(X)$ is a injective complex, and so it is exact. We have the following commutative diagram;

$$\begin{array}{ccccc}
0 & \longrightarrow & X & \xrightarrow{i} & E(X) \\
& & \downarrow id_x & \searrow \phi & \\
& & X & &
\end{array}$$

such that $\phi i = id_x$. Therefore X is a direct summand of $E(X)$. So X is an injective complex and hence it is exact. Similarly, if $\frac{Y}{X} \in \mathcal{X}$, then we can prove that X is a direct summand of Y . □

Theorem 2.26 *Let \mathcal{X} be a class under extensions, quotients and direct limits, then every complex B has an \mathcal{X} – injective envelope.*

Proof We know that B has an injective envelope E . Let $S = \{A : B \subseteq A \subseteq E \text{ and } \frac{A}{B} \text{ is an } \mathcal{X} \text{ – complex}\} \neq \emptyset$. Let S' be an ascending chain. Then $\frac{\cup_{N_i \in S'} N_i^k}{B^k} = \cup_{N_i \in S'} \frac{N_i^k}{B^k} = \lim \frac{N_i^k}{B^k}$ is in the class \mathcal{X} . So S has a maximal element, say T . We shall prove that T is an \mathcal{X} – injective complex. It is enough to show that any exact sequence $0 \longrightarrow T \longrightarrow Y \longrightarrow C \longrightarrow 0$ with $C \in \mathcal{X}$ -complex is split. We have the following commutative diagram;

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & T & \longrightarrow & Y & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \alpha & & \downarrow \beta & & \\
 0 & \longrightarrow & T & \xrightarrow{i} & E(T) & \xrightarrow{\pi} & \frac{E(T)}{T} & \longrightarrow & 0
 \end{array}$$

Let $\beta(C) = \frac{K}{T}$. So by $\frac{K}{T} \cong \frac{\frac{K}{B}}{\frac{T}{B}}$, we say that $\frac{K}{B}$ is an \mathcal{X} – complex. Since T is a maximal element of S , $\beta(C) = 0$, and hence $\pi\alpha(Y) = \frac{\alpha(Y)+T}{T} = 0$, and so $\alpha(Y) \subseteq T$. Therefore $0 \longrightarrow T \longrightarrow Y \longrightarrow C \longrightarrow 0$ is split exact and hence T is an \mathcal{X} – injective complex. Moreover T is an \mathcal{X} – injective preenvelope of B by the following diagram;

$$\begin{array}{ccccc}
0 & \longrightarrow & B & \longrightarrow & T \\
& & \downarrow & \nearrow & \\
& & Y & &
\end{array}$$

where Y is an \mathcal{X} -injective complex and $\frac{T}{B} \in \mathcal{X}$ and also if,

$$\begin{array}{ccccc}
0 & \longrightarrow & B & \xrightarrow{\alpha} & T \\
& & \downarrow \alpha & \nearrow \beta & \\
& & T & &
\end{array}$$

where $\beta\alpha = \alpha$, then β is $1 - 1$ since $B \subset^{ess} T$, and so $\beta(T) \cong T$. If $\beta(T) \neq T$, then $\beta(T)$ is an \mathcal{X} -injective complex such that $B \subseteq \beta(T) \subset T \subseteq E$. So we have $0 \longrightarrow T \xrightarrow{\beta} T \longrightarrow \frac{T}{\beta(T)} \longrightarrow 0$ split exact sequence and $\beta(T) \subset^{ess} T$, so $\beta(T) = T$. \square

Theorem 2.27 *X is essentially contained in a minimal \mathcal{X} -injective complex X' with $X \subset^{ess} X'$.*

Proof We know that every complex has an injective envelope. Let $S = \{A : X \subseteq A \subseteq E \text{ and } A \text{ } \mathcal{X}\text{-injective complex}\} \neq \emptyset$ and S' be a descending chain of S . We will show that $\cap_{A_\alpha \in S'} \{A_\alpha : \alpha \in I\}$ is an \mathcal{X} -injective complex. Using this pushout diagram we have the following diagram where C with \mathcal{X} -complex,

$$\begin{array}{ccccccc}
0 & \longrightarrow & \cap A_\alpha & \xrightarrow{\beta} & Y & \longrightarrow & C \longrightarrow 0 \\
& & \downarrow \theta_\alpha & & \downarrow \phi & & \downarrow \\
0 & \longrightarrow & A_\alpha & \xrightarrow{\gamma_\alpha} & B_\alpha & \longrightarrow & C \longrightarrow 0
\end{array}$$

Then the bottom row is split exact. So $0 \longrightarrow \cap A_\alpha \longrightarrow \cap B_\alpha \longrightarrow C \longrightarrow 0$ is split exact. We have the following diagram;

$$\begin{array}{ccccccc}
0 & \longrightarrow & \cap A_\alpha & \xrightarrow{\beta} & Y & \longrightarrow & C \longrightarrow 0 \\
& & \downarrow & & \downarrow \phi & & \downarrow \\
0 & \longrightarrow & \cap A_\alpha & \xrightarrow{\gamma} & \cap B_\alpha & \longrightarrow & C \longrightarrow 0
\end{array}$$

By five lemma ϕ is an isomorphism. So $0 \longrightarrow \cap A_\alpha \longrightarrow Y \longrightarrow C \longrightarrow 0$ is split exact. So S has a minimal element, say X' . \square

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